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A characterization of the rational vertex operator algebra $V_{\mathbb{Z}\alpha}^+$: II

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Abstract

A characterization of vertex operator algebra V_L^+ for any rank one positive definite even lattice L is given in terms of dimensions of homogeneous subspaces of small weights. This result reduces the classification of rational vertex operator algebras of central charge 1 to the characterization of three vertex operator algebras in the E-series of central charge one.

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1 Introduction

It is a well known conjecture in the theory of vertex operator algebra that any rational vertex operator algebras with central charge $c = 1$ is isomorphic to V_L, or V_L^+ or $V_{\mathbb{Z}\alpha}^G$ where L is a rank one positive definite even lattice, $(\alpha, \alpha) = 2$ and $G = A_4, S_4, A_5$ is a subgroup of $SO(3)$ in the E-series (cf. [K]). The characterization of V_L for any positive definite even lattice was established in [DM] in terms the rank of V_1, central charge and effective central charge. A characterization of $V_{\mathbb{Z}\beta}^+$ for $(\beta, \beta) = 4$ in terms of $\dim V_2$ is given in [ZD], [DJ1]. Characterization for all $V_{\mathbb{Z}\beta}^+$ with $(\beta, \beta)/2$ not being a perfect square in terms of dimensions of V_i for $i \le 4$ is obtained in [DJ2]. In this paper we characterize $V_{\mathbb{Z}\beta}$ with $(\beta, \beta)/2$ being a perfect square by dimensions of V_i for $i \le 4$. It remains the characterization of $V_{\mathbb{Z}\alpha}^G$ for $G = A_4, S_4$ and A_5 for completing the classification of rational vertex operator algebras with $c = 1$.

There are two major differences between $V_{\mathbb{Z}\beta}^+$ and $V_{\mathbb{Z}\alpha}^G$. The first one is that $V_{\mathbb{Z}\beta}^+$ is the fixed points of rational vertex operator algebra $V_{\mathbb{Z}\beta}$ under an order two automorphism and $V_{\mathbb{Z}\alpha}^G$ is the fixed points of rational vertex operator algebra $V_{\mathbb{Z}\alpha}$ under a nonabelian group. The rationality and classification of irreducible modules of $V_{\mathbb{Z}\alpha}^G$ have not been achieved although the automorphism groups of $V_{\mathbb{Z}\alpha}^G$ are known [DG], [DGR]. But this difference is not our concern in this paper. The second difference comes from the dimensions of weight 4 subspaces: $\dim(V_{\mathbb{Z}\beta}^+)_4 \ge 3$ and $\dim(V_{\mathbb{Z}\alpha}^G)_4 = 2$. This difference inspires us to characterize $V_{\mathbb{Z}\beta}^+$ in terms of dimensions of V_i for $i \le 4$ in [DJ2] and this paper. So one

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natural assumption for the vertex operator algebra V with $c = 1$ discussed in this paper is $\dim V_4 \geq 3$.

Although the rank of $L = \mathbb{Z}\beta$ is one, the vertex operator algebra V_L^+ is still a hard object to study. As the weight one subspace is zero and weight two subspace is spanned by the Virasoro vector, one can hardly use any results from the Lie algebra or Griess algebra to obtain useful information. On the other hand, the structure and representation theory of vertex operator algebras V_L^+ and its subalgebra $M(1)^+$ have been studied extensively in [A1], [A2], [AD], [ADL], [DN1], [DN2] and [DN3]. As in [DJ2] we use results from these papers to find a vertex operator subalgebra of V isomorphic to $M(1)^+$. But the situation is much more complicated as $(\beta, \beta)/2$ is a perfect square. We need different ideas and methods. It is well known that $M(1)^+$ in the rank one case is generated by the Virasoro vector ω and a highest weight vector J of weight 4. The main property of J is the following: $J_3J = x + aJ$ for some $x \in L(1, 0)$ which is the vertex operator subalgebra generated by ω and some nonzero $a \in \mathbb{C}$. It turns out that searching for such J in an abstract vertex operator algebra satisfying certain assumptions is a very difficult task and involves delicate use of the fusion rules for the vertex operator algebra $L(1, 0)$ [M], [DJ1].

We should point out that in the characterization of the lattice vertex operator algebras V_L for a positive definite even lattice L we need to use extra assumptions on the C_2-cofiniteness and the effective central charge (see [DLM1] for the reason). But we do not need the C_2-cofiniteness and the effective cental charge being one in the characterization of $V_{\mathbb{Z}\beta}^+$ with $(\beta, \beta)/2 > 2$ not being a perfect square [DJ2]. The situation for $(\beta, \beta)/2$ being a perfect square is totally different. Although the effective central charge never plays any role in this paper, the C_2-cofiniteness does. During the search for vector J in V we cannot avoid to use the modular invariance result from [Z] where the C_2-cofiniteness is assumed. This is not surprising as defining effective central charges requires the C_2-cofiniteness [DM] and the conjecture on rational vertex operator algebras with central charge 1 is not true without assuming the effective central charge is also one (see [ZD]).

We refer the readers to [AP] and [X] for the related work.

This paper is organized as follows. We review the fusion rules for the vertex operator algebra $L(1, 0)$ from [M] and [DJ1] in Section 2. We also present some results concerning Zhu algebra [Z] and calculations on J in $M(1)^+$. In Sections 3 and 4 we search for a highest weight vector J' of weight 4 in an abstract vertex operator V such that $J'_3J' = x + aJ'$ for some $x \in L(1, 0)$ and a nonzero $a \in \mathbb{C}$. If the space A_4 of highest weight vectors of weight 4 is one dimensional, it is trivial to find such J'. If $\dim A_4 \geq 2$ this is highly nontrivial. It is proved first that such J' exists if $\dim A_4 = 2$. Then it is shown that if $\dim A_4 \geq 2$ then $\dim A_4 = 2$. The fusion rules of $L(1, 0)$ is used heavily here. The modular invariance of trace functions also plays a role in this part. Section 5 is devoted to the proof that the vertex operator subalgebra U of V generated by ω and J' is isomorphic to $M(1)^+$. Section 6 gives the main theorem: V is isomorphic to $V_{\mathbb{Z}\beta}^+$ such that $(\beta, \beta) = 2k^2$ for some $k > 1$. A major step in this section is to show that V is a completely reducible $M(1)^+$-module with the help of fusion rules for $M(1)^+$ and $L(1, 0)$.

2 Preliminaries

In this section we recall the fusion rules for the Virasoro vertex operator algebra $L(1,0)$ from [M] and [DJ1] and for the vertex operator algebra $M(1)^+$ of central charge 1 from [A1]. We also discuss various results on the generator J of $M(1)^+$ following [DN1].

Let $L(c,h)$ be the highest weight irreducible module for the Virasoro algebra with central charge c and highest weight h. Then $L(c,0)$ is a vertex operator algebra and each $L(c,h)$ is an irreducible module for $L(c,0)$. In this paper we are mainly concerned with $L(1,0)$ and its irreducible modules. First, we have from [M] and [DJ1] (also see [RT]):

Theorem 2.1. *We have*

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\,L(1,n^2)\end{pmatrix} = 1, \quad k\in\mathbb{Z}_+,\ |n-m|\le k\le n+m, \tag{2.1}$$

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k^2) \\ L(1,m^2)\,L(1,n^2)\end{pmatrix} = 0, \quad k\in\mathbb{Z}_+,\ k<|n-m| \text{ or } k>n+m, \tag{2.2}$$

where $n,m\in\mathbb{Z}_+$. For $n\in\mathbb{Z}_+$ such that $n\neq p^2$ for all $p\in\mathbb{Z}_+$, we have

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,n) \\ L(1,m^2)\,L(1,n)\end{pmatrix} = 1, \tag{2.3}$$

$$\dim I_{L(1,0)}\begin{pmatrix} L(1,k) \\ L(1,m^2)\,L(1,n)\end{pmatrix} = 0, \tag{2.4}$$

for $k\in\mathbb{Z}_+$ such that $k\neq n$.

Recall the Heisenberg vertex operator algebra $M(1)$ constructed from a d-dimensional vector space and its subalgebra $M(1)^+$ from [FLM]. Based on the classification of irreducible modules for $M(1)^+$ [DN1], [DN3], the fusion rules for $M(1)^+$ have been obtained in [A1] and [ADL]. Here is the result when $d=1$.

Theorem 2.2. *Let M,N and T be irreducible $M(1)^+$-modules. If $M=M(1,\lambda)$ such that $\lambda\neq 0$, then*

$$\dim I_{M(1)^+}\begin{pmatrix} T \\ M\,N\end{pmatrix} = 0 \ \text{ or } \ 1$$

and

$$\dim I_{M(1)^+}\begin{pmatrix} T \\ M\,N\end{pmatrix} = 1$$

if and only if (N,T) is one of the following pairs:

$$(M(1)^{\pm}, M(1,\mu))(\lambda^2=\mu^2),\ (M(1,\mu),M(1,\nu)),\ (\nu^2=(\lambda\pm\mu)^2),$$

$$(M(1)(\theta)^{\pm}, M(1)(\theta)^{\pm}),\ (M(1)(\theta)^{\pm}, M(1)(\theta)^{\mp}).$$

For the purpose of later discussion we need to study $M(1)^+$ more. From now on we assume $d = 1$. Recall from [DN1] that

$$J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \frac{3}{2}h(-2)^2\mathbf{1} \in M(1)^+$$

is a primary vector of weight 4 in $M(1)^+$. Then $M(1)^+$ is generated by ω and J. Let $M^{(4)}$ be the $L(1,0)$-submodule of $M(1)^+$ generated by J. Then $J_7 J = 54\mathbf{1}$. Moreover, as a module for $L(1,0)$,

$$M(1)^+ = \bigoplus_{n\geq 0} L(1,(2n)^2)$$

where $L(1,0) \cong M^{(4)}$. Following [Z] we set

$$u * v = \mathrm{Res}_z \left(\frac{(1+z)^{\mathrm{wt}(u)}}{z^{1+n}} Y(u,z)v \right)$$

for homogeneous $u, v \in W$ where W is any vertex operator algebra. Then

$$J * J = \sum_{j=0}^{4} \binom{4}{j} J_{j-1}J = u^{(0)} + v^{(0)},$$

where $u^{(0)} \in L(1,0)$ and $v^{(0)} \in M^{(4)}$.

Lemma 2.3. *We have*

$$u^{(0)} \in p(\omega) + O(L(1,0)),\ v^{(0)} \in q(\omega)J + (L(-1) + L(0))M^{(4)}$$

where

$$p(x) = x\left(\frac{1816}{35}x^3 - \frac{212}{5}x^2 + \frac{89}{10}x - \frac{27}{70}\right),$$
$$q(x) = -\frac{314}{35}x^2 + \frac{89}{14}x - \frac{27}{70}$$

and the product is $*$.

Proof: By [DN1], we have in $M(1)^+$

$$u^{(0)} + v^{(0)} \equiv p(\omega) + q(\omega)J \bmod O(M(1)^+).$$

Since $J * J \in \sum_{i=0}^{8} M(1)^+_i$ we see that

$$u^{(0)} \in p_1(\omega) + O(L(1,0)),\ v^{(0)} \in q_1(\omega)J + (L(-1) + L(0))M^{(4)}$$

where $p_1(x)$ is a polynomial with degree ≤ 4 and $q_1(x)$ is a polynomial with degree ≤ 2. Note that $O(L(1,0)), (L(-1)+L(0))M^{(4)} \subseteq O(M(1)^+)$. As in [DN1] we apply the identity $J * J = u^{(0)} + v^{(0)}$ to the irreducible $A(M(1)^+)$-modules to conclude that $p_1(x) = p(x)$ and $q(x) = q_1(x)$, as desired. □

Lemma 2.4. *$p(x)$ has no non-zero integer roots.*

Proof: It is easy to check that

$$x_1 = 0,\ x_2 = \frac{1}{4},\ x_3 = \frac{515 + \sqrt{167161}}{1816},\ x_4 = \frac{515 - \sqrt{167161}}{1816}$$

are all the roots of $p(x)$. $\square$

The following lemma will be used later.

Lemma 2.5. *In $M(1)^+$, we have*

$$\begin{aligned}
J_3J &= -72L(-4)\mathbf{1} + 336L(-2)^2\mathbf{1} + \lambda J,\\
J_2J &= u^1 + \lambda\frac{1}{2}L(-1)J,\\
J_1J &= u^2 + \lambda(\frac{28}{75}L(-2)J + \frac{23}{300}L(-1)^2J),\\
J_0J &= u^3 + \lambda(\frac{14}{75}L(-3)J + \frac{14}{75}L(-2)L(-1)J - \frac{1}{300}L(-1)^3J),
\end{aligned}$$

for some $u^i \in L(1,0), i = 1,2,3,\ 0 \neq \lambda \in \mathbb{C}$.

Proof: We first deal with J_3J. Note that $J_iJ \in L(1,0) \oplus M^{(4)}$ for $i \geq 0$. Then there exist $\lambda_1, \lambda_2, \lambda \in \mathbb{C}$ such that $J_3J = \lambda_1 L(-4)\mathbf{1} + \lambda_2 L(-2)^2\mathbf{1} + \lambda J$. Using the commutator formula

$$[L(m), J_n] = [3(m+1) - n]J_{m+m}$$

for $m, n \in \mathbb{Z}$ one can check that

$$(L(-4)\mathbf{1}, J_3J) = (\mathbf{1}, L(4)J_3J) = 12 \times 54,$$

$$(L(-2)^2\mathbf{1}, J_3J) = (\mathbf{1}, L(2)^2J_3J) = 24 \times 54.$$

On the other hand,

$$(L(-4)\mathbf{1}, J_3J) = (L(-4)\mathbf{1}, \lambda_1 L(-4)\mathbf{1} + \lambda_2 L(-2)^2\mathbf{1}) = 5\lambda_1 + 3\lambda_2,$$

$$(L(-2)^2\mathbf{1}, J_3J) = (L(-2)^2\mathbf{1}, \lambda_1 L(-4)\mathbf{1} + \lambda_2 L(-2)^2\mathbf{1}) = 3\lambda_1 + \frac{2}{4}\lambda_2.$$

This implies that $\lambda_1 = -72$ and $\lambda_2 = 336$. It follows from the proof of Theorem 4.9 of [DJ2] that $\lambda \neq 0$. One can also verify J_3J directly with a long computation.

We now prove other relations. We may assume that

$$\begin{aligned}
J_2J &= u^1 + \mu_0 L(-1)J,\\
J_1J &= u^2 + \mu_1 L(-2)J + \mu_2 L(-1)^2J,\\
J_0J &= u^3 + \mu_3 L(-3)J + \mu_4 L(-2)L(-1)J + \mu_5 L(-1)^3J
\end{aligned}$$

where $u^i \in L(1,0), i=1,2,3,\ \mu_j \in \mathbb{C}, j=0,1,\cdots,5$. Then

$$(J_2J, L(-1)J) = (L(1)J_2J, J) = 4\lambda(J,J),$$

$$(J_1J, L(-2)J) = (L(2)J_1J, J) = 8\lambda(J,J),$$
$$(J_1J, L(-1)^2J) = (L(1)^2J_1J, J) = 20\lambda(J,J).$$

We also have

$$(L(-1)J, L(-1)J) = (L(1)L(-1)J, J) = 8(J,J),$$
$$(L(-2)J, L(-2)J) = (L(2)L(-2)J, J) = \frac{33}{2}(J,J),$$
$$(L(-2)J, L(-1)^2J) = (L(1)^2L(-2)J, J) = 24(J,J),$$
$$(L(-1)^2J, L(-1)^2J) = (L(1)^2L(-1)^2J, J) = 144(J,J).$$

These relations yield the following linear equations

$$\mu_0 = \frac{1}{2}\lambda$$

and

$$\begin{cases} 33\mu_1 + 48\mu_2 = 16\lambda \\ 24\mu_1 + 144\mu_2 = 20\lambda \end{cases}$$

with solutions

$$\mu_1 = \frac{28}{75}\lambda,\ \ \mu_2 = \frac{23}{300}\lambda.$$

Similarly,

$$(J_0J, L(-3)J) = (L(3)J_0J, J) = 12\lambda(J,J),$$
$$(J_0J, L(-2)L(-1)J) = (L(1)L(2)J_0J, J) = 36\lambda(J,J),$$
$$(J_0J, L(-1)^3J) = (L(1)^3J_0J, J) = 120\lambda(J,J),$$
$$(L(-3)J, L(-3)J) = 26(J,J),\ \ (L(-3)J, L(-2)L(-1)J) = 40(J,J),$$
$$(L(-3)J, L(-1)^3J) = 96(J,J),\ \ (L(-2)L(-1)J, L(-2)L(-1)J) = 164(J,J),$$
$$(L(-2)L(-1)J, L(-1)^3J) = 624(J,J),\ \ (L(-1)^3J, L(-1)^3J) = 4320(J,J).$$

Then we get the following linear system

$$\begin{cases} 26\mu_3 + 40\mu_4 + 96\mu_5 = 12\lambda \\ 40\mu_3 + 164\mu_4 + 624\mu_5 = 36\lambda \\ 96\mu_3 + 624\mu_4 + 4320\mu_5 = 120\lambda \end{cases}$$

with solutions

$$\mu_3 = \mu_4 = \frac{14}{75}\lambda,\ \ \mu_5 = -\frac{1}{300}\lambda.$$

The lemma follows. $\square$

3 Search for vector J: I

In the following discussion throughout the paper, we always assume that V is a simple rational and C_2-cofinite vertex operator algebra of central charge 1 satisfying the following conditions:

(1) V is a sum of highest weight modules of $L(1,0)$.

(2) $V=\oplus_{n=0}^{\infty}V_n,\ V_0=\mathbb{C}\mathbf{1},\ V_1=0,\ \dim V_2=\dim V_3=1,\ \dim V_4\geq 3.$

(3) The weights of all the primary vectors in V are perfect squares.

Remark 3.1. *As we mentioned in the introduction already, we have dealt with the case that there exists at least one primary vector in V whose weight is not a perfect square in [DJ2].*

In this section and the next we look for a primary vector J' of weight 4 in V such that J' satisfies all relations given in Lemma 2.5 for J. This will help us to show that the vertex operator subalgebra U generated by ω and J' is isomorphic to $M(1)^+$ with identifying J with J'. It turns out that finding such J' is highly nontrivial and an explicit construction of intertwining operators for $L(1,0)$ involving modules $L(1,4)$ and $L(1,0)$ plays crucial role in the proof.

Let X^1 and X^2 be two subsets of V. Set

$$X^1\cdot X^2=span\{x_ny|x\in X^1,y\in X^2,n\in\mathbb{Z}\}.$$

We have the following lemmas from [DJ2] (see also [DJ1]).

Lemma 3.2. *V is a completely reducible module for the Virasoro algebra $L(1,0)$.*

Lemma 3.3. *Let $u^1,u^2\in V$ be two primary vectors. Let U^1 and U^2 be two $L(1,0)$-submodules of V generated by u^1 and u^2 respectively. Then*

$$U^1\cdot U^2=span\{L(-m_1)\cdots L(-m_s)u^1_nu^2|m_1,\cdots,m_s\in\mathbb{Z}_+,n\in\mathbb{Z}\}.$$

For $m\geq 1$, set

$$A_{m^2}=\{v\in V_{m^2}|L(n)v=0,n\geq 1\}.$$

Elements in A_{m^2} are called primary vectors. Since $V_1=0$, it follows that $m\geq 2$. It is obvious that V carries a non-degenerate symmetric bilinear form $(\cdot,\cdot)$ such that $(\mathbf{1},\mathbf{1})=1$ ([FHL], [L]). By the assumption (2), $\dim V_2=\dim V_3=1$ and $A_4\neq 0$. Let J' be a non-zero primary vector of weight 4. We may assume that

$$(J',J')=54. \tag{3.1}$$

By Theorem 2.1 and Lemma 2.5 there exists a primary vector u of weight 4 such that

$$J'_3J'=-72L(-4)\mathbf{1}+336L(-2)^2\mathbf{1}+27u.$$

It is possible that

$$u=\frac{1}{27}J'_3J'+\frac{8}{3}L(-4)\mathbf{1}-\frac{112}{9}L(-2)^2\mathbf{1}$$

is zero.

Here is the main result in this section.

Proposition 3.4. *The u is a nonzero primary vector of weight 4. In particular, $J_3'J'$ does not lie in $L(1,0)$.*

It is not easy to prove this result. We need several lemmas. Let $V^{(4)}$ be the $L(1,0)$-submodule of V generated by J'. Then $V^{(4)}$ is isomorphic to $L(1,4)$.

Lemma 3.5. *Let U be the vertex operator subalgebra of V generated by ω and J'. If $u \in \mathbb{C}J'$, then U is linearly spanned by*

$$L(-m_1)\cdots L(-m_s)J'_{-n_1}\cdots J'_{-n_t}\mathbf{1}$$

where $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$, $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$, $s,t \geq 0$.

Proof: First note that the subspace of U linearly spanned by $J'_{n_1}\cdots J'_{n_t}\mathbf{1}$ with $n_i \in \mathbb{Z}$ is invariant under the action of $L(m), m \geq -1$. Secondly, we have

$$[J'_m, J'_n] = \sum_{i=0}^{\infty} \begin{pmatrix} m \\ i \end{pmatrix} (J'_iJ')_{m+n-i}. \tag{3.2}$$

Since $\mathrm{wt}(J') = 4$, we have $\mathrm{wt}(J'_iJ') = 7 - i \leq 7$, for $i \geq 0$.

As $u \in \mathbb{C}J'$, $J'_3J \in L(1,0) + M^{(4)}$. It follows from Theorem 2.1 and Lemma 3.3 that $J'_iJ' \in L(1,0) \oplus V^{(4)}$ for $i \geq 0$. The lemma is clear. $\square$

Lemma 3.6. *Let U be the vertex operator subalgebra of V generated by ω and J'. If $u \in \mathbb{C}J'$, then the Zhu algebra $A(U)$ is linearly spanned by*

$$\{[\omega]^s * [J']^t,\ s,t \geq 0\}.$$

Proof: The proof of Theorem 3.5 in [DN1] for the spanning set of $A(M((1)^+)$ works here. $\square$

Lemma 3.7. *If $u = 0$, then*

$$J' * J' \equiv p(\omega) + O(L(1,0)) \equiv p(\omega) + O(V).$$

Proof: By Theorem 2.1 and Lemma 3.5 we see that $J' * J' \in L(1,0)$. Since $J_7J = 54\mathbf{1}$ and $J'_7J' = 54\mathbf{1}$, it follows from Theorem 2.1 again that if $u = 0$ then

$$J' * J' = \sum_{j=0}^{4} \begin{pmatrix} 4 \\ j \end{pmatrix} J'_{j-1}J' = u^{(0)},$$

where $u^{(0)}$ is the same as in Section 2. Then the lemma follows from Lemma 2.3. $\square$

Lemma 3.8. *Let U be the vertex operator subalgebra of V generated by ω and J'. If $u = 0$, then $U = V$.*

Proof: By Lemma 3.6, $A(U)$ is commutative. Suppose that $U \neq V$. Let $m \in \mathbb{Z}_+$ be the smallest positive integer such that $A_{m^2} \nsubseteq U$. Then $m \geq 2$. Note that V/U is a U-module with the minimal weight m^2. Let $0 \neq u^{(m)} \in A_{m^2}$, $u^{(m)} \notin U$. Then $u^{(m)} + U$ generates a U-submodule of V/U and let W be the irreducible quotient. We denote the image of $u^{(m)} + U$ by $v^{(m)}$. Then W has lowest weight m^2 and

$$J'_i v^{(m)} = 0 \tag{3.3}$$

for $i \geq 4$. By Lemmas 2.4 and 3.7 we know that $J'_3 J'_3 v^{(m)} = p(L(0))v^{(m)} \neq 0$. As a result,

$$J'_3 v^{(m)} \neq 0. \tag{3.4}$$

Since W is completely reducible as an $L(1,0)$-module, $v^{(m)}$ generates an irreducible highest weight $L(1,0)$-submodule $W^{(m^2)}$ of W isomorphic to $L(1,m^2)$ with highest weight vector $v^{(m)}$. By the skew symmetry, we have

$$J'_{-2}J' = \frac{1}{2}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{L(-1)^i}{i!}J'_{-2+i}J'.$$

Then by Theorem 2.1, Lemma 3.5, (3.3) and the assumption that $u = 0$, we see that

$$V^{(4)} \cdot V^{(4)} \cong L(1,0) \oplus a_4 L(1,16). \tag{3.5}$$

$$V^{(4)} \cdot W^{(m^2)} \cong W^{(m^2)} \oplus b_{m+1}L(1,(m+1)^2) \oplus b_{m+2}L(1,(m+2)^2), \tag{3.6}$$

where $a_4, b_{m+1}, b_{m+2} \in \mathbb{Z}$ are nonnegative.

Let $\mathcal{P}$ be the projection from $V^{(4)} \cdot W^{(m^2)}$ to $W^{(m^2)}$, then $\mathcal{I}(u,z)v = \mathcal{P} \cdot Y(u,z)v$ for $u \in V^{(4)}$, $v \in W^{(m^2)}$ is an intertwining operator of type

$$\left(\begin{array}{c} L(1,m^2) \\ V^{(4)} \quad L(1,m^2) \end{array}\right).$$

Let $(V_L^+, Y(\cdot,z))$ be the rank one rational vertex operator algebra with $L = \mathbb{Z}\beta$ such that $(\beta,\beta) = 2m^2$. Set

$$E^{(m)} = e^{\beta} + e^{-\beta} \in V_L^+.$$

Then

$$J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \frac{3}{2}h(-2)^2\mathbf{1} \in M(1)^+ \subset V_L^+,$$

where $h = \frac{1}{\sqrt{2}m}\beta$. Let $M^{(m^2)}$ be the irreducible $L(1,0)$-modules in V_L^+ generated by $E^{(m)}$. From the construction of V_L^+ we know

$$M^{(4)} \cdot M^{(m^2)} \cong M^{(m^2)} \oplus L(1,(m+1)^2) \oplus L(1,(m+2)^2), \tag{3.7}$$

$$M^{(4)} \cdot M^{(4)} \cong L(1,0) \oplus M^{(4)} \oplus L(1,16). \tag{3.8}$$

Let $\mathcal{Q}$ be the projection from $M^{(4)} \cdot M^{(m^2)}$ to $M^{(m^2)}$. Then $\mathcal{I}'(u,z)v = \mathcal{Q} \cdot Y(u,z)v$ for $u \in M^{(4)}$, $v \in M^{(m^2)}$ is an intertwining operator of type

$$\left(\begin{array}{c} M^{(m^2)} \\ M^{(4)} \quad M^{(m^2)} \end{array} \right).$$

Let σ be the $L(1,0)$-module isomorphism from $M^{(4)} \oplus M^{(m^2)}$ to $V^{(4)} \oplus W^{(m^2)}$ such that

$$\sigma(J) = J', \ \sigma(E^{(m)}) = v^{(m)}.$$

By Theorem 2.1, for $u \in M^{(4)}, v \in M^{(m^2)}$,

$$\mathcal{I}(\sigma u, z)(\sigma v) = c\sigma(\mathcal{I}(u,z)v),$$

for some $c \in \mathbb{C}$. By (3.4), $c \neq 0$. Note that

$$(J_7'J')_{-1}v^{(m)} = \sum_{i=0}^{\infty}(-1)^i \left(\begin{array}{c} 7 \\ i \end{array} \right) (J_{7-i}'J_{-1+i}' + J_{6-i}'J_i')v^{(m)}.$$

By (3.6), for $i \geq 0$, we have

$$J_{-1+i}'v^{(m)}, \ J_{7-i}'J_{-1+i}'v^{(m)}, \ J_i'v^{(m)}, \ J_{6-i}'J_i'v^{(m)} \in W^{(m^2)}.$$

On the other hand, we have

$$(J_7J)_{-1}E^{(m)} = \sum_{i=0}^{\infty}(-1)^i \left(\begin{array}{c} 7 \\ i \end{array} \right) (J_{7-i}J_{-1+i} + J_{6-i}J_i)E^{(m)}.$$

By (3.7), for $i \geq 0$,

$$J_{-1+i}E^{(m)}, \ J_{7-i}J_{-1+i}E^{(m)}, \ J_iE^{(m)}, \ J_{6-i}J_iE^{(m)} \in M^{(m^2)}.$$

Thus we have

$$(J_7'J')_{-1}v^{(m)} = c^2\sigma((J_7J)_{-1}E^{(m)}).$$

Note that

$$(J_7'J')_{-1}v^{(m)} = 54v^{(m)} = \sigma((J_7J)_{-1}E^{(m)}).$$

We deduce that $c^2 = 1$. Then we may assume that $c = 1$. If $c = -1$, replace J' by $-J'$.

Since

$$(J_3'J')_3v^{(m)} = \sum_{i=0}^{\infty}(-1)^i \left(\begin{array}{c} 3 \\ i \end{array} \right) (J_{3-i}'J_{3+i}' + J_{6-i}'J_i')v^{(m)},$$

$$(J_3J)_3E^{(m)} = \sum_{i=0}^{\infty}(-1)^i \left(\begin{array}{c} 3 \\ i \end{array} \right) (J_{3-i}J_{3+i} + J_{6-i}J_i)E^{(m)},$$

it follows that

$$(J_3'J')_3v^{(m)} = \sigma((J_3J)_3E^{(m)}). \tag{3.9}$$

Recall from Lemma 2.5 that

$$J_3 J = x + \lambda J$$

for some $x \in L(1,0)$ and nonzero $\lambda \in \mathbb{C}$. Suppose that

$$J_3' J' = x' + y',$$

where $x' \in L(1,0)$, $y' \in V^{(4)}$. Then by the fact that $(J, J) = (J', J')$, we have

$$x' = \sigma(x).$$

Then by (3.9), for any $w \in M^{(m^2)}$,

$$y_3'(\sigma(w)) = \sigma(y_3 w).$$

A straightforward computation shows that $J_3 E^{(m)} \neq 0$. This implies that $y' \neq 0$, a contradiction with (3.5). This proves that $U = V$. □

We are now in a position to prove Proposition 3.4.

Proof: Suppose that $u = 0$, then $U = V$ by Lemma 3.8 and

$$V^{(4)} \cdot V^{(4)} \cong L(1,0) \oplus a_4 L(1,16),$$

where $a_4 \in \mathbb{N}$. If $a_4 = 0$, then

$$U = L(1,0) \oplus V^{(4)}.$$

Let W be a module for the Virasoro algebra with central charge c such that $W = \bigoplus_{n\in\mathbb{C}} W_n$ where W_n is the eigenspace for $L(0)$ with eigenvalue n and is finite-dimensional. We define the q-graded dimension of W as

$$\dim_q W = q^{-c/24} \sum_{n\in\mathbb{C}} (\dim W_n) q^n.$$

Denote by $L(c,h)$ the unique irreducible highest weight module for the Virasoro algebra with central charge $c \in \mathbb{C}$ and highest weight $h \in \mathbb{C}$. Then

$$\dim_q L(1,h) = \begin{cases} \frac{1}{\eta(q)}(q^{n^2/4} - q^{(n+2)^2/4}), & \text{if } h = \frac{1}{4}n^2, n \in \mathbb{Z} \\ \frac{1}{\eta(q)} q^h, & \text{otherwise.} \end{cases}$$

(cf. [KR]) where

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Then

$$Z_V(\tau) = ch_q L(1,0) + ch_q L(1,4) = \frac{1 - q + q^4 - q^9}{\eta(q)}$$

where $q = e^{2\pi i \tau}$ and τ is a complex variable in the upper half plane. We sometimes abuse the notation and also denote $\eta(q)$ by $\eta(\tau)$. Since V is rational and C_2-cofinite we use the

modular invariance result given in [Z] to assert that $Z_V(\frac{-1}{\tau})$ should have a q-expansion. It is well known that $\eta(\frac{-1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$. Thus

$$Z_V(\frac{-1}{\tau}) = \frac{1 - e^{-2\pi i \frac{1}{\tau}} + e^{-2\pi i \frac{4}{\tau}} - e^{-2\pi i \frac{9}{\tau}}}{(-i\tau)^{\frac{1}{2}}\eta(\tau)}$$

which clearly does not have a q-expansion. This gives a contradiction.

So there exists a non-zero primary vector $u^{(4)}$ of weight 16 such that

$$u^{(4)} = a_1 J'_{-9}J' + x,$$

for some $0 \neq a_1 \in \mathbb{C}$, $x \in L(1,0) \oplus V^{(4)}$. Then by Lemma 3.5 and (3.2), U is linearly spanned by

$$L(-m_1)\cdots L(-m_s)\mathbf{1},\ L(-p_1)\cdots L(-p_s)J'_{-n_1}\cdots J'_{-n_t}J'_{-9}J',\ L(-p_1)\cdots L(-p_s)J' \quad (3.10)$$

where $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$, $n_1 \geq n_2 \geq \cdots \geq n_t \geq 9$, $p_1 \geq p_2 \geq \cdots \geq p_s \geq 1$. By Theorem 2.1 and (3.2), in (3.10) we may assume that $n_t \geq 17$.

It is easy to see from (3.10) that there is no non-zero primary vector of weight 25. If there is no non-zero primary vector of weight 36, then by Theorem 2.1, $J'_n J'_{-9}J' \in L(1,0) \oplus V^{(4)} \oplus V^{(16)}$ for $n \in \mathbb{Z}$ where $V^{(16)}$ is the $L(1,0)$-module generated by $u^{(4)}$. This forces that

$$V \cong L(1,0) \oplus V^{(4)} \oplus V^{(16)}.$$

The same proof as above gives a contradiction. So there exists a non-zero primary vector $u^{(6)}$ of weight 36 such that

$$u^{(6)} = a_2 J'_{-17}J'_{-9}J' + x,$$

for some $0 \neq a_2 \in \mathbb{C}$, $x \in L(1,0) \oplus V^{(4)} \oplus V^{(16)}$. Continuing the process, we deduce that V is linearly spanned by

$$L(-m_1)\cdots L(-m_s)\mathbf{1},\ L(-n_1)\cdots L(-n_t)J',\ L(-n_1)\cdots L(-n_t)J'_{-8r-1}\cdots J'_{-9}J',$$

where $m_1 \geq m_2 \geq \cdots m_s \geq 2$, $n_1 \geq n_2 \geq \cdots n_t \geq 1$, $r \geq 1, s, t \geq 0$ and as a vector space

$$V \cong \bigoplus_{r=0}^{\infty} L(1,(2r)^2). \quad (3.11)$$

Then

$$Z_V(\tau) = ch_q V = \frac{\sum_{n\geq 0}(-q)^{n^2}}{\eta(q)} = \frac{1}{2\eta(q)} + \frac{\theta_{0,1}(1,q)}{2\eta(q)}$$

where

$$\theta_{0,1}(1,q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}.$$

is the theta function. It is well known that $\frac{\theta_{0,1}(1,q)}{2\eta(q)}$ is a modular function over a congruence subgroup of $SL(2,\mathbb{Z})$ and $\frac{1}{2\eta(q)}$ is a modular form of weight $-\frac{1}{2}$. On the other hand, $Z_V(\tau)$ is a component of a vector-valued modular function (cf. [Z], [KM], [DM]). This implies that $\frac{1}{2\eta(q)}$ is a component of vector-valued modular function over a congruence subgroup of $SL(2,\mathbb{Z})$. This is obviously impossible. So V can not be the form of (3.11). $\square$

4 Search for vector J: II

In this section we prove that there exists a non-zero primary vector X of weight 4 such that $(X,X)\neq 0$ and

$$X_3X = v + cX,$$

for some $v\in L(1,0)$ and $0\neq c\in\mathbb{C}$. Recall that A_4 is the space of primary vectors in V_4. If $\dim A_4 = 1$, then by Proposition 3.4, the J' given in Section 3 is the desired element. From now on we assume that $\dim A_4\geq 2$. We will prove that the result is true if $\dim A_4 = 2$ and then show that $\dim A_4$ must be 2.

Assume that $\dim A_4 = 2$. Clearly, there exists $K\in A_4$ such that

$$(K,K) = 54,\ (J',K) = 0. \tag{4.1}$$

It follows from Lemma 2.5 that the projection of J_3J to $L(1,0)$ is

$$X^0 = -72L(-4)\mathbf{1} + 336L(-2)^2\mathbf{1}. \tag{4.2}$$

Then by the fusion rules of $L(1,0)$ (see Theorem 2.1 and (4.1)) we have

$$J'_3J' = X^0 + a_1J' + b_1K,\ K_3K = X^0 + a_2J' + b_2K$$

for some $a_i, b_i\in\mathbb{C}$ with $i = 1,2$. If $b_1 = 0$ or $a_2 = 0$, then by Lemma 3.4, either J' or K is the desired element $X\in A_4$. So in the following discussion we assume that $b_1\neq 0$, $a_2\neq 0$.

From (4.1) and Theorem 2.1 we see that $K_iJ' = 0$ for all $i > 3$. Using the skew-symmetry yields $J'_3K = K_3J'$. Since

$$(J'_3K, J') = (K, J'_3J'),\ (J'_3K, K) = (J', K_3K)$$

we see that

$$J'_3K = K_3J' = b_1J' + a_2K. \tag{4.3}$$

Lemma 4.1. *If* $\dim A_4 = 2$, *there exists* $X\in A_4$ *such that*

$$X_3X = c_1X^0 + c_2X, \tag{4.4}$$

for some $c_1, c_2\in\mathbb{C}$, *where* X^0 *is defined as (4.2).*

Proof: For $\mu_1,\mu_2\in\mathbb{C}$, we have

$$\begin{aligned}&(\mu_1J' + \mu_2K)_3(\mu_1J' + \mu_2K)\\ =&(\mu_1^2+\mu_2^2)X^0 + (\mu_1^2a_1 + 2\mu_1\mu_2b_1 + \mu_2^2a_2)J' + (\mu_1^2b_1 + 2\mu_1\mu_2a_2 + \mu_2^2b_2)K.\end{aligned}$$

By the assumption that $b_1\neq 0$ and $a_2\neq 0$, we may assume that $\mu_1\neq 0$. Then $X = \mu_1J' + \mu_2K$ satisfies (4.4) for some $c_1, c_2\in\mathbb{C}$ if and only if μ_1 and μ_2 satisfy

$$\frac{\mu_1^2a_1 + 2\mu_1\mu_2b_1 + \mu_2^2a_2}{\mu_1} = \frac{\mu_1^2b_1 + 2\mu_1\mu_2a_2 + \mu_2^2b_2}{\mu_2}.$$

That is,

$$a_2(\frac{\mu_2}{\mu_1})^3 + (2b_1 - b_2)(\frac{\mu_2}{\mu_1})^2 + (a_1 - 2a_2)\frac{\mu_2}{\mu_1} - b_1 = 0. \tag{4.5}$$

It is clear that the above equation has solution $\frac{\mu_2}{\mu_1} \in \mathbb{C}$. The lemma follows. $\square$

In the following two lemmas we do not need to assume that A_4 is 2-dimensional.

Lemma 4.2. *Let $X \in A_4$ be such that $X_3X = \mu X^0 + \nu X$ for some $\mu, \nu \in \mathbb{C}$. If $\mu = 0$, then $\nu = 0$.*

Proof: Suppose that $\mu = 0$, then $X_3X = \nu X$. By Theorem 2.1 and Lemma 2.5 one deduces that

$$X_2X = \frac{1}{2}\nu L(-1)X, \tag{4.6}$$

$$X_1X = \frac{28}{75}\nu L(-2)X + \frac{23}{300}\nu L(-1)^2X, \tag{4.7}$$

$$X_0X = \frac{14}{75}\nu L(-3)X + \frac{14}{75}\nu L(-2)L(-1)X - \frac{1}{300}\nu L(-1)^3X. \tag{4.8}$$

Since $\mu = 0$, it follows that $(X, X) = 0$. Thus $X_iX = 0$ for $i \geq 4$. In particular, $(X_4X)_2X = 0$.

On the other hand by (4.6)-(4.8), we have

$$\begin{aligned}
&(X_4X)_2X \\
=&\sum_{i=0}^{4}(-1)^i \begin{pmatrix} 4 \\ i \end{pmatrix} (X_{4-i}X_{2+i} - X_{6-i}X_i)X \\
=&-5X_4X_2X + 4X_5X_1X - X_6X_0X \\
=&-5\frac{1}{2}\nu X_4L(-1)X + 4X_5(\frac{28}{75}\nu L(-2)X + \frac{23}{300}\nu L(-1)^2X) \\
&-X_6(\frac{14}{75}\nu L(-3)X + \frac{14}{75}\nu L(-2)L(-1)X - \frac{1}{300}\nu L(-1)^3X).
\end{aligned}$$

Using the commutator formula

$$[L(m), X_n] = [3(m+1) - n]X_{m+n}$$

for $m, n \in \mathbb{Z}$ and the fact that $X_iX = 0$ for $i \geq 4$ we check that

$$X_4L(-1)X = 4X_3X,\ X_5L(-2)X = 8X_3X,\ X_5L(-1)^2X = 20X_3X,$$

$$X_6L(-3)X = 12X_3X,\ X_6L(-2)L(-1)X = 36X_3X,\ X_6L(-1)^3X = 120X_3X.$$

Then we deduce that

$$(X_4X)_2X = -\frac{12}{25}\nu^2X.$$

This proves that $\nu = 0$. $\square$

Lemma 4.3. *There are no non-zero elements $X^1, X^2 \in A_2$ such that*

$$(X^1, X^1) = 0,\ X^1_3 X^1 = 0,\ X^2_3 X^1 = \mu X^{(0)} \tag{4.9}$$

for some nonzero $\mu \in \mathbb{C}$.

Proof: Suppose that there are non-zero elements $X^1, X^2 \in A_4$ such that (4.9) holds. Note that $Y(X^1, z)X^1 \neq 0$ [DL]. Let N^i be the irreducible $L(1,0)$-modules generated by X^i for $i = 1, 2$. By Theorem 2.1, Lemma 3.3 and the skew-symmetry, $N^1 \cdot N^1 \cong L(1, 16)$. In particular,

$$X^1_{-9} X^1 \neq 0,\ X^1_i X^1 = 0,\ i \geq -8.$$

Since $X^2_6 X^1 \in V_1 = 0$ we see that

$$\begin{aligned}
(X^2_6 X^1)_0 X^1 &= \sum_{i=0}^{6} (-1)^{i+1} \begin{pmatrix} 6 \\ i \end{pmatrix} X^1_{6-i} X^2_i X^1 \\
&= \sum_{i=0}^{6} \sum_{j=0}^{7-i} \sum_{s=0}^{8} (-1)^{i+j+s+1} \begin{pmatrix} 6 \\ i \end{pmatrix} \begin{pmatrix} -8 \\ j \end{pmatrix} \begin{pmatrix} 8 \\ s \end{pmatrix} X^2_{i+j+s} X^1_{6-i-j-s} X^1 \\
&= \sum_{i=0}^{6} \begin{pmatrix} 6 \\ i \end{pmatrix} \begin{pmatrix} -8 \\ 7-i \end{pmatrix} K_{15} X_{-9} X \\
&= 8 X^2_{15} X^1_{-9} X^1 \\
&= 0.
\end{aligned}$$

This shows that $X^2_{15} X^1_{-9} X^1 = 0$. Then we have

$$\begin{aligned}
&(X^2_7 X^1)_{-1} X^1 \\
&= \sum_{i=0}^{7} (-1)^i \begin{pmatrix} 7 \\ i \end{pmatrix} X^1_{6-i} X^2_i X \\
&= \sum_{i=0}^{7} \sum_{j=0}^{7-i} \sum_{s=0}^{8} (-1)^{i+j+s} \begin{pmatrix} 7 \\ i \end{pmatrix} \begin{pmatrix} -8 \\ j \end{pmatrix} \begin{pmatrix} 8 \\ s \end{pmatrix} X^2_{i+j+s} X^1_{6-i-j-s} X^1 \\
&= -\sum_{i=0}^{7} \begin{pmatrix} 7 \\ i \end{pmatrix} \begin{pmatrix} -8 \\ 7-i \end{pmatrix} X^2_{15} X^1_{-9} X^1 \\
&= 0.
\end{aligned}$$

But by (4.9) we know that the projection of $Y(u, z)v$ for $u \in N^2$ and $v \in N^1$ to $L(1,0)$ is a nonzero intertwining operator of type $\begin{pmatrix} L(1,0) \\ L(4,0) \quad L(4,0) \end{pmatrix}$. In particular, $X^2_7 X^1 = (X^2, X^1)\mathbf{1}$ is nonzero. This gives a contradiction and the proof is complete. □

Lemma 4.4. *Assume that $\dim A_4 = 2$, then there exists $X \in A_4$ such that $(X, X) \neq 0$ and*

$$X_3 X = c_1 X^{(0)} + c_2 X,$$

for some $0 \neq c_1, 0 \neq c_2 \in \mathbb{C}$.

Proof: By lemma 4.1, there exists $X = \mu_1 J' + \mu_2 K \in A_4$ such that μ_1, μ_2 satisfy (4.5) and

$$X_3 X = c_1 X^{(0)} + c_2 X,$$

for some $c_1, c_2 \in \mathbb{C}$. Note that (4.5) has three solutions. If for one solution, $c_1 \neq 0$, by Lemma 3.4, $c_2 \neq 0$. Then the lemma holds.

Suppose that for all the three solutions of (4.5), $c_1 = 0$. Then $c_2 = 0$ by Lemma 4.2. Let $\nu = \dfrac{\mu_2}{\mu_1}$, then

$$1 + \nu^2 = 1,\ a_1 + 2b_1\nu + a_2\nu^2 = 0,\ b_1 + 2a_2\nu + b_2\nu^2 = 0.$$

If $\nu = \sqrt{-1}$, then

$$a_1 + 2\sqrt{-1}b_1 - a_2 = 0,\ b_1 + 2\sqrt{-1}a_2 - b_2 = 0.$$

If $\nu = -\sqrt{-1}$, then

$$a_1 - 2\sqrt{-1}b_1 - a_2 = 0,\ b_1 - 2\sqrt{-1}a_2 - b_2 = 0.$$

So if (4.5) has different solutions, then $b_1 = a_2 = 0$, a contradiction with the assumption. This deduces that all the solutions of (4.5) are $\nu_1 = \nu_2 = \nu_3 = \sqrt{-1}$ or $\nu_1 = \nu_2 = \nu_3 = -\sqrt{-1}$. Without loss of generality, we assume that $\nu_1 = \nu_2 = \nu_3 = \sqrt{-1}$. Using the relation between roots and coefficients of the equation (4.5) we see that $-\sqrt{-1} = \frac{b_1}{a_2}$. Consequently,

$$b_1 = -a_2\sqrt{-1},\ a_1 = -a_2,\ b_2 = a_2\sqrt{-1}.$$

We deduce that

$$J'_3 J' = X^0 + a_1(J' + \sqrt{-1}K),$$
$$K_3 K = X^0 - a_1(J' + \sqrt{-1}K).$$

By Proposition 3.4, $a_1 \neq 0$. We may assume that $\mu_1 = 1$. Then $X = J' + \sqrt{-1}K$. Let $K' = J' - \sqrt{-1}K$. Then we have from (4.3) that

$$(X, X) = 0,\ X_3 X = 0,\ (K', K') = 0,\ K'_3 K' = 4a_1 X,\ K'_3 X = 2X^0.$$

This contradicts Lemma 4.3. □

We next establish that $\dim A_4 \geq 2$ implies $\dim A_4 = 2$.

Lemma 4.5. *If* $\dim A_4 \geq 2$*, then* $\dim A_4 = 2$.

Proof: Let $X^1, \cdots, X^s$ be a basis of A_4 such that

$$(X^i, X^j) = 2\delta_{ij},\ i, j = 1, 2, \cdots, s. \tag{4.10}$$

Recall X^0 from (4.2). Then from the discussion on $J'_3 J'$ we have

$$X^i_3 X^j = \frac{1}{27}\delta_{ij} X^0 + \sum_{k=1}^{s} a^k_{ij} X^k$$

for some $a_{ij}^k \in \mathbb{C}$, $i, j, k = 1, 2, \cdots, s$. The invariant property

$$(X_3^i X^j, X^k) = (X_3^j X^i, X^k) = (X^j, X_3^i X^k)$$

then gives

$$a_{ij}^k = a_{ji}^k = a_{ik}^j \tag{4.11}$$

for $i, j, k = 1, 2, \cdots, s$. For $1 \leq k \leq s$ we define matrix $A^{(k)} = (a_{ij}^k)_{i,j=1}^s$.

Using the relation $J_4 J = 216L(-3)\mathbf{1}$ and (4.10) we see that $X_4^i X^j = \delta_{ij} 8L(-3)\mathbf{1}$ for $i, j \in \{1, 2, \cdots, s\}$. This implies for any k that

$$(X_4^i X^j)_2 X^k = -64\delta_{ij} X^k. \tag{4.12}$$

By Lemma 7.1

$$(X_4^i X^j)_2 X^k = u - 2\sum_{r=1}^s \sum_{l=1}^s a_{jk}^r a_{ir}^l X^l + \frac{114}{75} \sum_{r=1}^s \sum_{l=1}^s a_{ik}^r a_{jr}^l X^l + a\delta_{jk} X^i + b\delta_{ik} X^j$$

for some $a, b \in \mathbb{C}$ where

$$\begin{aligned} u =& \frac{1}{2} a_{jk}^i L(-1) X_4^i X^i + a_{ik}^j \left(\frac{28}{75} L(-2) + \frac{11}{30} L(-1)^2\right) X_5^j X^j \\ & - \frac{197}{150} a_{jk}^i L(-1) X_4^j X^j + \frac{1}{27}\left(\frac{114}{75} a_{ik}^j - 2a_{jk}^i\right) X^{(0)}. \end{aligned}$$

Applying $L(1)$ to $X_4^i X^i = 8L_{-3}\mathbf{1}$ produces $X_5^i X^i = 16L_{-2}\mathbf{1}$ for $i = 1, 2, \cdots, s$. Since $(X_4^i X^j)_2 X^k = -64\delta_{ij} X^k \in A_4$ and $u \in L(1, 0)$ we see that $u = 0$ and

$$-2\sum_{r=1}^s \sum_{l=1}^s a_{jk}^r a_{ir}^l X^l + \frac{114}{75} \sum_{r=1}^s \sum_{l=1}^s a_{ik}^r a_{jr}^l X^l + a\delta_{jk} X^i + b\delta_{ik} X^j = -64\delta_{ij} X^k.$$

Comparing the coefficients of X^l of both sides and varying i, j we have for all l, k that

$$-\frac{36}{75} A^{(k)} A^{(k)} = -(a + b) E_{kk} - 64I. \tag{4.13}$$

$$\frac{114}{75} A^{(k)} A^{(l)} - 2A^{(l)} A^{(k)} = -aE_{lk} - bE_{kl} - 64\delta_{kl} I. \tag{4.14}$$

$$\frac{114}{75} A^{(l)} A^{(k)} - 2A^{(k)} A^{(l)} = -aE_{kl} - bE_{lk} - 64\delta_{lk} I, \tag{4.15}$$

where I is the identity matrix and $E_{pq} = (e_{ij})_{i,j=1}^s$ such that $e_{ij} = \delta_{ip}\delta_{jq}$. Then we deduce that for $1 \leq k, l \leq s$, $k \neq l$,

$$\frac{11}{75} A^{(k)} A^{(l)} = \frac{1}{144}[(25b + 19a)E_{lk} + (19b + 25a)E_{kl}]. \tag{4.16}$$

Now suppose that $s \geq 3$. For $1 \leq k \leq s$, denote by $r(A^{(k)})$ the rank of $A^{(k)}$. By (4.16), $r(A^{(k)}) \leq s-1$. It follows from (4.13) that $a+b=-64$ and $\frac{9}{75}A^{(k)}A^{(k)} = 16(I-E_{kk})$. Using $\frac{11\times 9}{75}(A^{(1)}A^{(1)})A^{(2)} = \frac{11\times 9}{75}A^{(1)}(A^{(1)}A^{(2)})$ gives

$$176\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a^2_{21} & a^2_{22} & \cdots & a^2_{2,s-1} & a^2_{2s} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a^2_{s1} & a^2_{s2} & \cdots & a^2_{s,s-1} & a^2_{ss} \end{bmatrix} = \frac{1}{16}\begin{bmatrix} d_1a^1_{12} & d_2a^1_{11} & 0 & \cdots & 0 \\ d_1a^1_{22} & d_2a^1_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ d_1a^1_{s2} & d_2a^1_{s1} & 0 & \cdots & 0 \end{bmatrix} \tag{4.17}$$

where $d_1 = 25b+19a$ and $d_2 = 19b+25a$. Similarly, by the fact that

$$(A^{(1)}A^{(2)})A^{(2)} = A^{(1)}(A^{(2)}A^{(2)}),$$

we have

$$176\begin{bmatrix} a^1_{11} & 0 & a^1_{13} & \cdots & a^1_{1s} \\ a^1_{21} & 0 & a^1_{23} & \cdots & a^1_{2s} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a^1_{s1} & 0 & a^1_{s3} & \cdots & a^1_{ss} \end{bmatrix} = \frac{1}{16}\begin{bmatrix} d_2a^2_{21} & d_2a^2_{22} & \cdots & d_2a^2_{s2} \\ d_1a^2_{11} & d_1a^2_{12} & \cdots & d_1a^2_{1s} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{4.18}$$

By (4.17), $a^2_{ij} = 0$ for $i \geq 2, j > 2$. Then by (4.11), $a^2_{ji} = 0$ with $i \geq 2, j > 2$. Using (4.17) again asserts

$$d_2a^1_{j1} = 0,\ \ j = 3, 4, \cdots, s. \tag{4.19}$$

Assume $d_2 \neq 0$ and $d_1 \neq 0$. It follows from (4.19) that $a^1_{j1} = 0$ for $j \geq 3$. Using (4.17) and (4.18) also gives $a^1_{11} = a^1_{12} = 0$. This implies that $a^k_{11} = 0$ for all k and

$$X^1_3X^1 = \frac{1}{27}X^{(0)}. \tag{4.20}$$

This contradicts Proposition 3.4. So $d_1 = 0$ or $d_2 = 0$. If $d_1 = 25b+19a = 0$ then $d_2 = 19b+25a \neq 0$ as $a+b=-64$. We have $a^1_{1j} = 0$ for $j = 1$ or $j > 2$. By (4.18), $a^1_{12} = a^1_{21} = 0$. So we have (4.20) again. Similarly if $d_2 = 0$ then $d_1 \neq 0$, $a^1_{1j} = 0$ for all j and (4.20) holds. The proof is complete. $\square$

5 The subalgebra $M(1)^+$ of V

In this section, we prove that there is a vertex operator subalgebra U of V isomorphic to $M(1)^+$. By Lemmas 4.4 and 4.5, there exists $J' \in A_4$ such that

$$J'_3J' = X^0 + cJ' \tag{5.1}$$

for some $0 \neq c \in \mathbb{C}$. Let U be the subalgebra of V generated by ω and J'. Recall that $V^{(4)}$ is the irreducible $L(1,0)$-submodule of V generated by J'. From the skew-symmetry

$$Y(J', z)J' = e^{L(-1)z}Y(J', -z)J'$$

we see that

$$J'_{-2}J' = -J'_{-2}J' + \sum_{j=1}^{9}(-1)^{j+1}\frac{1}{j!}L(-1)^j J'_{-2+j}J'.$$

This together with Theorem 2.1 deduces that

$$V^{(4)} \cdot V^{(4)} \subseteq L(1,0) \oplus V^{(4)} \oplus L(1,16). \tag{5.2}$$

The proof of the following lemma is similar to that of Proposition 3.4.

Lemma 5.1. *The vertex operator subalgebra U is not equal to the whole algebra V.*

Recall Lemma 2.3 and $J_3J = X^{(0)} + \lambda J$ from Lemma 2.5. The following is an analogue to Lemma 2.3.

Lemma 5.2. *We have $J' * J' = u^{(0)} + \frac{c}{\lambda}v^{(0)}$ in U where*

$$u^{(0)} \in p(\omega) + O(L(1,0)),\ v^{(0)} \in q(\omega)J' + (L(-1) + L(0))V^{(4)}$$

where $p(x)$ and $q(x)$ are defined in Lemma 2.5.

Proof: First we define four projections:

$$p_1 : M(1)^+ \to L(1,0),\ p_2 : M(1)^+ \to M^{(4)},\ p_1' : U \to L(1,0),\ p_2' : U \to V^{(4)}.$$

Then $p_1Y(u,z)v$ and $p_2Y(u,z)v$ for $u, v \in M^{(4)}$ define two intertwining operators of types $\begin{pmatrix} L(1,0) \\ M^{(4)}\ M^{(4)} \end{pmatrix}$ and $\begin{pmatrix} M^{(4)} \\ M^{(4)}\ M^{(4)} \end{pmatrix}$, respectively. Similarly, $p_1'Y(u,z)v$ and $p_2'Y(u,z)v$ for $u, v \in V^{(4)}$ define two intertwining operators of types $\begin{pmatrix} L(1,0) \\ V^{(4)}\ V^{(4)} \end{pmatrix}$ and $\begin{pmatrix} V^{(4)} \\ V^{(4)}\ V^{(4)} \end{pmatrix}$, respectively. Identify $L(1,0)$ in both $M(1)^+$ and U. Since $J_3J = X^{(0)} + \lambda J$ and $J_3'J' = X^{(0)} + cJ'$, the result follows from Theorem 2.1 and Lemma 2.3 immediately. □

By Lemmas 5.1 and 4.5 there exists $2 \leq k \in \mathbb{Z}_+$ such that $U_m = V_m$ for $m < k^2$ and $U_{k^2} \neq V_{k^2}$. Take $F \in A_{k^2} \notin U$. As in the proof of Lemma 3.8, V/U is a U-module with the minimal weight k^2 and $F^{(k)} + U$ generates a U-submodule W of V/U. Let $\bar{W}$ be the irreducible quotient of W. We denote the image of $F + U$ in $\bar{W}$ by a. Note that $\bar{W}$ has the lowest weight k^2 and a generates an irreducible $L(1,0)$-submodule $W^{(k^2)}$ of $\bar{W}$. By Lemma 3.6, $A(U)$ is commutative. It follows that the lowest weight subspace is one-dimensional. Then

$$J_3'a \in \mathbb{C}v,\ J_n'a = 0,$$

for $n \geq 4$. Since $p(L(0))a \neq 0$ (see Lemma 2.4) we immediately deduce from Lemma 5.2 that $J_3'a \neq 0$.

Let V_L^+ be the rational vertex operator algebra associated to the definite even lattice $L = \mathbb{Z}\alpha$ such that $(\alpha, \alpha) = 2k^2$. Set

$$E = e^{\alpha} + e^{-\alpha} \in V_L^+,$$

$$J = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \frac{3}{2}h(-2)^2\mathbf{1} \in V_L^+,$$

where $h = \frac{1}{\sqrt{2}k}\alpha$. We identify the Virasoro vertex operator subalgebra $L(1,0)$ in V and V_L^+. Let $M^{(k^2)}$ be the irreducible $L(1,0)$-module in V_L^+ generated by E. Then there exists an $L(1,0)$-module isomorphisms σ from $M^{(4)}$ to $V^{(4)}$ and $M^{(k^2)}$ to $W^{(k^2)}$ such that $\sigma J = J'$ and $\sigma E = a$.

Let P and P' be the projections of V_L^+ to $M^{(k^2)}$ and $\bar{W}$ to $W^{(k^2)}$ respectively. Then $I(u, z)w = P \cdot Y(u, z)w$ and $I'(u', z)w' = P' \cdot Y(u', z)w'$ for $u \in M^{(4)}, w \in M^{(k^2)}$, $u' \in V^{(4)}, w' \in W^{(k^2)}$ are intertwining operators of types $\left(\begin{array}{c} M^{(k^2)} \\ M^{(4)}\, M^{(k^2)} \end{array} \right)$ and $\left(\begin{array}{c} W^{(k^2)} \\ V^{(4)}\, W^{(k^2)} \end{array} \right)$, respectively. Let Q and Q' be the projections of V_L^+ to $M^{(4)}$ and V to $V^{(4)}$, respectively. Then $\mathcal{I}(u, z)w = Q \cdot Y(u', z)w'$ and $\mathcal{I}'(u', z)w'$ for $u, w \in M^{(4)}$, $u', w' \in V^{(4)}$ are intertwining operators of type $\left(\begin{array}{c} M^{(4)} \\ M^{(4)}\, M^{(4)} \end{array} \right)$ and $\left(\begin{array}{c} V^{(4)} \\ V^{(4)}\, V^{(4)} \end{array} \right)$, respectively.

It follows from Theorem 2.1 that $J'_n a \in W^{(k^2)}$ and

$$(J'_3 J')_n a = \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} 3 \\ i \end{array} \right) (J'_{3-i} J'_{n+i} + J'_{3+n-i} J'_i) a \in W^{(k^2)}$$

for $n \geq 3 - 2k$. Using the proof of Lemma 4.5 and Lemma 4.7 in [DJ2] we have

Lemma 5.3.

$$I'(\sigma(u), z)\sigma(w) = \sigma(I(u, z)w), \quad \mathcal{I}'(\sigma(u), z)\sigma(v) = \sigma(\mathcal{I}'(u, z)v)$$

for $u, v \in M^{(4)}, w \in W^{(k^2)}$. In particular, let $\lambda \in \mathbb{C}$ be such that

$$J_3 J = X^{(0)} + \lambda J,$$

then

$$J'_3 J' = X^{(0)} + \lambda J'.$$

Recall that $J'_n w \in W^{(k^2)}$ for $n \geq 3 - 2k$. That is, $J'_n w$ is a linear combination of vectors of form $L(-n_1) \cdots L(-n_k)w$ with $3 - n = -n_1 - n_2 \cdots - n_k$. Using the relation $[L(m), J'_n] = (3(m + 1) - n)J'_{m+n}$ we see that if $n \geq 15 - 2k$ then

$$(J'_{-9} J')_n a = \sum_{i=0}^{\infty} (-1)^i \left(\begin{array}{c} -9 \\ i \end{array} \right) (J'_{-9-i} J'_{n+i} + J'_{-9+n-i} J'_i) a \in W^{(k^2)}.$$

A proof similar to that of Lemma 4.10 in [DJ2] gives that

Lemma 5.4. *There exist a non-zero primary element u^4 of weight 16 in V_L^+ and a non-zero primary element v^4 of weight 16 in U such that the isomorphism σ from $L(1,0)\bigoplus M^{(4)}$ to $L(1,0)\bigoplus V^{(4)}$ can be extended to an isomorphism σ from $L(1,0)\bigoplus M^{(4)}\bigoplus M^{(16)}$ to $L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}$ such that $\sigma(u^4)=v^4$ and for $n\in\mathbb{Z}$, $u,v\in L(1,0)\bigoplus M^{(4)}$,*

$$(\sigma u)_n(\sigma v)=\sigma(u_nv), \tag{5.3}$$

where $M^{(4^2)}$ is the irreducible $L(1,0)$-submodule of V_L^+ generated by u^4 and $V^{(4^2)}$ is the irreducible $L(1,0)$-submodule of U generated by v^4. In particular, for $m,n\in\mathbb{Z}$,

$$[J'_m,J'_n]=\sum_{j=0}^{\infty}\left(\begin{array}{c} m \\ j \end{array}\right)(\sigma(J_jJ))_{m+n-j}. \tag{5.4}$$

We can now have a "nicer" spanning set for U.

Lemma 5.5. *U is linearly spanned by*

$$L(-m_s)\cdots L(-m_1)\mathbf{1},\ L(-n_s)\cdots L(-n_1)J',\ L(-n_s)\cdots L(-n_1)J'_{-8t-1}\cdots J'_{-17}J'_{-9}J'$$

where $m_s\geq m_{s-1}\geq\cdots m_1\geq 2$, $n_s\geq n_{s-1}\geq\cdots n_1\geq 1$, $t\geq 1$, $s\geq 0$.

Proof: By Lemma 3.5 and (5.1) U is linearly spanned by

$$L(-m_s)\cdots L(-m_1)\mathbf{1},\ L(-n_s)\cdots L(-n_1)J',\ L(-n_s)\cdots L(-n_1)J'_{-p_t}\cdots J'_{-p_1}J'$$

where $m_s\geq m_{s-1}\geq\cdots\geq m_1\geq 2$, $n_s\geq n_{s-1}\geq\cdots\geq n_1\geq 1$, $p_t\geq p_{t-1}\geq\cdots\geq p_1\geq 1$, $s,t\geq 0$.

It follows from (5.1) and Theorem 2.1 that

$$L(-n_s)\cdots L(-n_1)J'_{-p_1}J'\in L(1,0)\bigoplus V^{(4)}$$

for $p_1\leq 8$. So we can assume that $p_1\geq 9$. Using (5.2) gives

$$L(-n_s)\cdots L(-n_1)J'_{-p_1}J'\in L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}$$

for $p_1\geq 9$. By Lemma 5.4, $V^{(4)}\cdot V^{(4)}\cong L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}$ and there is a non-zero primary vector $v^{(16)}$ of weight 16 in $V^{(4)}\cdot V^{(4)}$ such that $v^{(16)}=x+aJ'_{-9}J'$ for some $x\in L(1,0)\bigoplus V^{(4)}$ and $0\neq a\in\mathbb{C}$. So we may assume that $p_1=9$.

If there exists a non-zero primary vector u of weight 25 then $u=u^1+aJ'_{-6}J'_{-9}J'$ for some $u^1\in L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}$ and $0\neq a\in\mathbb{C}$. Note that $J'_{-6}J', J'_jJ'\in L(1,0)\bigoplus V^{(4)}$ for $j\geq 0$. So

$$J'_{-6}J'_{-9}J'=J'_{-9}J'_{-6}J'+\sum_{j=0}^{\infty}\left(\begin{array}{c} -6 \\ j \end{array}\right)(J'_jJ')_{-15-j}J'\in L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}.$$

This proves that there is no non-zero primary vector of weight 25. By Theorem 2.1 again

$$L(-n_s)\cdots L(-n_1)J'_{-p_2}J'_{-p_1}J' \in L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}$$

for $p_2 < 17$ and

$$L(-n_s)\cdots L(-n_1)J'_{-p_2}J'_{-p_1}J' \in L(1,0)\bigoplus V^{(4)}\bigoplus V^{(16)}\bigoplus L(1,36)$$

for $p_2 \geq 17$. So we may assume that $p_2 = 17$. Continuing in this way gives the result. □

The following is the main result of this section.

Theorem 5.6. *There is a vertex operator algebra isomorphism σ from $M(1)^+$ to U such that $\sigma\omega = \omega$ and $\sigma(J) = J'$.*

Proof: Let u^4, v^4 and σ be the same as in Lemma 5.4. Then there exist $x^{(4)} \in L(1,0)\bigoplus M^{(4)}$ and $0 \neq a_1 \in \mathbb{C}$ such that

$$u^4 = a_1 J_{-9}J + x^{(4)},\ v^4 = a_1 J'_{-9}J' + \sigma(x^{(4)}).$$

Moreover, $(u,v) = (\sigma(u),\sigma(v))$ for $u, v \in L(1,0) + M^{(4)} + M^{(16)}$. From the construction of $M(1)^+$ [DG], there exists a non-zero primary element u^6 of weight 36 in $M(1)^+$ such that $J_{-17}J_{-9}J = u^6 + x^{(6)}$, where $x^{(6)} \in L(1,0)\bigoplus M^{(4)}\bigoplus M^{(16)}$. It is obvious that $(J_{-17}J_{-9}J, J_{-17}J_{-9}J) = (u^6,u^6) + (x^{(6)},x^{(6)})$. Set $M^{(0)} = V^{(0)} = L(1,0)$. Let P_i and Q_i be the projections of V_L^+ and U to $M^{(i)}$ and $V^{(i)}$, respectively for $i = 0,4,16$. Then $I^i(u,z)v = P_iY(u,z)v$ and $J^i(\sigma u,z)\sigma v = Q_iY(\sigma u,z)\sigma v$ for $u \in M^{(4)}$, $v \in M^{(16)}$ are intertwining operators of type $\begin{pmatrix} M^{(i)} \\ M^{(4)}\ M^{(16)} \end{pmatrix}$ and $\begin{pmatrix} V^{(i)} \\ V^{(4)}\ V^{(16)} \end{pmatrix}$ respectively for $i = 0,4,16$.

By Lemma 5.4 for $n \geq -16$,

$$\begin{aligned} J'_nJ'_{-9}J' &= \sum_{j=0}^{\infty}\begin{pmatrix} n \\ j \end{pmatrix}(J'_jJ')_{-9+n-j}J' + J'_{-9}J'_nJ' \\ &= \sum_{j=0}^{\infty}\begin{pmatrix} n \\ j \end{pmatrix}\sigma((J_jJ)_{-9+n-j}J) + \sigma(J_{-9}J_nJ) \\ &= \sigma(J_nJ_{-9}J). \end{aligned}$$

Note that $J_{-9}J \in L(1,0)\bigoplus M^{(4)} \oplus M^{(16)}$ from the structure of $M(1)^+$ and there exist $m,n \geq 3$ such that the projections of $J_mJ_{-9}J$ and $J_nJ_{-9}J$ to $M^{(4)}$ and $M^{(16)}$ are nonzero. We also know that $J'_{-9}J' \in L(1,0) \oplus V^{(4)} \oplus V^{(16)}$. It follows that

$$J^i(\sigma u,z)\sigma v = \sigma(I^i(u,z)v) \tag{5.5}$$

for $i = 2,4$ and $u \in M^{(4)}$ and $v \in M^{(16)}$. By Theorem 2.1 and (5.5), there exists a non-zero

primary vector v^6 of weight 36 in U such that $J'_{-17}J'_{-9}J' = v^6 + \sigma(x^{(6)})$. By (5.5) again

$$\begin{aligned}
&(J'_{-17}J'_{-9}J', J'_{-17}J'_{-9}J') = (J'_{-9}J', J'_{23}J'_{-17}J'_{-9}J') \\
&= \sum_{j=0}^{\infty} \left(\begin{array}{c} 23 \\ j \end{array} \right) (J'_{-9}J', (J'_jJ')_{6-j}J'_{-9}J') \\
&= \sum_{j=0}^{\infty} \left(\begin{array}{c} 23 \\ j \end{array} \right) (J_{-9}J, (J_jJ)_{6-j}J_{-9}J) \\
&= (J_{-17}J_{-9}J, J_{-17}J_{-9}J).
\end{aligned}$$

In particular, $(v^6, v^6) = (u^6, u^6)$.

Let $M^{(36)}$ be the irreducible $L(1,0)$-submodule of V_L^+ generated by u^6 and $V^{(36)}$ the irreducible $L(1,0)$-submodule of U generated by v^6. Then the $L(1,0)$-module isomorphism σ from $M^{(0)} + M^{(4)} + M^{(16)}$ to $V^{(0)} + V^{(4)} + V^{(16)}$ can be extended to the $L(1,0)$-module isomorphism σ from $M^{(0)} + M^{(4)} + M^{(16)} + M^{(36)}$ to $V^{(0)} + V^{(4)} + V^{(16)} + V^{(36)}$ such that

$$\sigma(u^6) = v^6, \ \sigma(J_{-17}J_{-9}J) = J'_{-17}J'_{-9}J'$$

and for $n \geq -24$,

$$\sigma(J_nJ_{-17}J_{-9}J) = J'_nJ'_{-17}J'_{-9}J'.$$

Similarly, we have for any $n \in \mathbb{Z}$,

$$\sigma(J_nJ_{-17}J_{-9}J) = J'_nJ'_{-17}J'_{-9}J'.$$

Continuing the above process and using Lemma 5.5, we deduce that as a vector space,

$$U \cong \bigoplus_{m=0}^{\infty} L(1, 4m^2)$$

and there is an $L(1,0)$-module isomorphism σ from $M(1)^+$ to U such that

$$\sigma(x) = x, \sigma(J) = J', \ \sigma(J_{-8m-1}\cdots J_{-17}J_{-9}J) = J'_{-8m-1}\cdots J'_{-17}J'_{-9}J'$$

for $x \in L(1,0)$and $m \geq 1$, and for $n \in \mathbb{Z}$

$$\sigma(J_nJ_{-8m-1}\cdots J_{-17}J_{-9}J) = J'_nJ'_{-8m-1}\cdots J'_{-17}J'_{-9}J'.$$

Then it follows from Theorem 5.7.1 in [LL] that σ is an isomorphism of vertex operator algebras. □

6 Identification of V with V_L^+

In this section, we will prove that V is isomorphic to the rational vertex operator algebra $V_{\mathbb{Z}\alpha}^+$ where $(\alpha, \alpha) = 2k^2$ and $k \geq 2$ is the smallest positive integer such that $U_{k^2} \neq V_{k^2}$.

By Theorem 5.6, U is a simple vertex operator subalgebra of V isomorphic to $M(1)^+$. Then the restriction of the non-degenerate invariant symmetric bilinear form on V to U is non-degenerate. We identify U with $M(1)^+$. Then

$$V = M(1)^+ \bigoplus (M(1)^+)^\perp, \tag{6.1}$$

where $(M(1)^+)^\perp$ is the orthogonal complement of $M(1)^+$ in V with respect to the bilinear form. Clearly, $(M(1)^+)^\perp$ is also an $M(1)^+$-module. Since V is rational, it follows that $(M(1)^+)^\perp \neq 0$. Then $(M(1)^+)^\perp \cong \bigoplus_{m\geq k} c_m L(1,m^2)$ for some $k \geq 2$ such that $c_k \neq 0$ and $c_m \in \mathbb{N}$. Then $(M(1)^+)^\perp = \sum_{m\geq k}(M(1)^+)^\perp_m$. It is obvious that

$$L(m)(M(1)^+)^\perp_{k^2} = J'_n(M(1)^+)^\perp_{k^2} = 0, J'_3(M(1)^+)^\perp_{k^2} \subseteq (M(1)^+)^\perp_{k^2} \tag{6.2}$$

for $m \geq 1$ and $n \geq 4$.

Lemma 6.1. *Let W be an $M(1)^+$-module such that W is a completely reducible $L(1,0)$-module. Let $v \in W$ be a non-zero primary element of weight n^2 ($n \geq 2$ is an integer) such that $J'_3 v \in \mathbb{C}v$ and $J'_m v = 0$ for $m \geq 4$. Then the $M(1)^+$-submodule N of W generated by v is irreducible.*

Proof: Since N is a completely reducible $L(1,0)$-module, it follows from Theorem 2.1 that

$$N = \bigoplus_{p=0}^{\infty} c_p L(1,(n+p)^2),$$

for some $c_p \in \mathbb{N}$, $p = 0,1,2,\cdots$. Let $\bar{N}$ be the irreducible quotient of N. Then (see [DG] and [DN1])

$$\bar{N} = \bigoplus_{p=0}^{\infty} L(1,(n+p)^2)$$

as an $L(1,0)$-module. So $c_p \geq 1, p = 0,1,2,\cdots$. Obviously, N is linearly spanned by

$$L(-m_s)\cdots L(-m_1)v,\ L(-n_s)\cdots L(-n_1)J'_{-p_t}\cdots J'_{-p_1}v$$

where $m_s \geq m_{s-1} \geq \cdots \geq m_1 \geq 1$, $n_s \geq n_{s-1} \geq \cdots \geq n_1 \geq 1$, $p_t \geq p_{t-1} \geq \cdots \geq p_1$, $s,t \geq 0$. By Theorem 2.1 and the fact that $J'_3 v \in \mathbb{C}v$, we may assume that $p_1 \geq 2n-2$ in (6.3). Using a similar proof given in Lemma 5.5 shows that N is, in fact, spanned by

$$L(-m_s)\cdots L(-m_1)v, \quad L(-n_s)\cdots L(-n_1)J'_{2-2(n+i)}\cdots J'_{2-2(n+1)}J'_{2-2n}v \tag{6.3}$$

where $m_s \geq m_{s-1} \geq \cdots m_1 \geq 1$, $n_s \geq n_{s-1} \geq \cdots n_1 \geq 1$, $i \geq 0$, $s \geq 0$. Let N^i denote the subspace of N spanned by the elements given in (6.3) for fixed i. Then each N^i is an $L(1,0)$-submodule of N and $\sum_i N_i = N$.

Let v^1 be a highest weight vector of weight $(n+1)^2$ in N. Then by (6.3)

$$v^1 = u^1 + aJ'_{2-2n}v$$

for some $u^1 \in L(1,n^2), 0 \neq a \in \mathbb{C}$. Thus $N^0 \cong L(1,n^2) \oplus L(1,(n+1)^2)$. Similarly, let v^2 be a highest weight vector of weight $(n+2)^2$ in N. Then

$$v^2 = u^2 + aJ'_{2-2(n+1)}J'_{2-2n}v$$

for some $u^2 \in N^0, 0 \neq a \in \mathbb{C}$ and $N^1 \cong L(1,n^2) \oplus L(1,(n+1)^2) \oplus L(1,(n+2)^2)$. Continuing in this way we show that $N^i \cong \oplus_{j=0}^{i+1} L(1,(n+j)^2)$ for all $i \geq 1$. This implies that $N \cong \bar{N}$ as $L(1,0)$-modules. Consequently, $N = \bar{N}$ is an irreducible $M(1)^+$-module. The proof is complete. □

For convenience, denote $(M(1)^+)^{\perp}_{k^2}$ by B_k. Let P^k be the $M(1)^+$-submodule of V generated by B_k.

Lemma 6.2. *The restriction of $(\cdot,\cdot)$ to P^k is still non-degenerate.*

Proof: By (6.1), the restriction of $(\cdot,\cdot)$ to B_k is non degenerate. By (6.2), B_k is a $\mathbb{C}[J'_3]$-module. Let $0 \subset W^1 \subset W^2 \subset \cdots \subset W^t = B_k$ be a chain of $\mathbb{C}[J'_3]$-modules such that W^{i+1}/W^i is irreducible where $W^0 = 0$. Let S^i be the $M(1)^+$-submodule of V generated by W^i. Then $S^t = P^k$ and S^{i+1}/S^i is an irreducible $M(1)^+$-module by Lemma 6.1. This implies that if $u \in P^k$ satisfying $L(m)u = J'_n u = 0$ for all $m \geq 1$ and $n \geq 4$ and $L(0)u \in \mathbb{C}u, J'_3 u \in \mathbb{C}u$ then $u \in B_k$.

Now let R be the radical of the restriction of the bilinear form to P^k. Then R is an $M(1)^+$-submodule of P^k. If $R \neq 0$, then R contains an irreducible $M(1)^+$-submodule whose intersection with B_k is nonzero. As a result the restriction of the bilinear form to B_k is degenerate, a contradiction. This completes the proof. □

Using Lemma 6.2 gives the following decomposition

$$V = (M(1)^+ \bigoplus P^k) \bigoplus (M(1)^+ \bigoplus P^k)^{\perp}.$$

Moreover, $(M(1)^+ \bigoplus P^k)^{\perp}$ is an $M(1)^+$-module. Then

$$(M(1)^+ \bigoplus P^k)^{\perp} = \bigoplus_{m \geq k_1} b_m L(1,m^2)$$

as an $L(1,0)$-module where $k_1 > k$ and $b_{k_1} \geq 1$. Similar to Lemma 6.2, the restriction of $(\cdot,\cdot)$ to the $M(1)^+$-module generated by $(M(1)^+ \bigoplus P^k)^{\perp}_{k_1^2}$ is non-degenerate. Continuing in this way we deduce the following lemma.

Lemma 6.3. *As an $M(1)^+$-module, V has the following submodule decomposition:*

$$V = M(1)^+ \bigoplus (\bigoplus_{i=0}^{\infty} P^{k_i})$$

where $k_0 = k < k_1 < k_2 < \cdots$ and P^{k_i} is the $M(1)^+$-submodule of V generated by

$$B_{k_i} = \{u \in V_{k_i^2} | L_m u = J'_n u = 0, m \geq 1, n \geq 4\}.$$

We can now prove the following lemma.

Lemma 6.4. *We have* $\dim B_{k_i} = 1$ *for* $i = 0, 1, 2, \cdots$. *Furthermore,* V *is a completely reducible* $M(1)^+$*-module.*

Proof: By Lemma 6.1 and Lemma 6.3, it is enough to prove that $\dim B_{k_i} = 1$, for $i \geq 0$. We only prove $\dim B_{k_0} = 1$ as the proof for other cases is similar. Suppose $\dim B_k \geq 2$. Then there exist $x^1, x^2 \in B_k$ such that $(x^i, x^i) = 0$ and $(x^1, x^2) = 1$ for $i = 1, 2$ (see the proof of Lemma 5.2 of [DJ2]), and

$$J_3' x^1 = (4k^4 - k^2)x^1. \tag{6.4}$$

Denote by M^i the irreducible $L(1,0)$-submodule of V generated by x^i respectively, $i = 1, 2$. Then $M^1 \cong M^2 \cong L(1, k^2)$. We first prove

Claim: $M^1 \cdot M^1 \cong L(1, 4k^2)$ is an irreducible $L(1,0)$-module.

Let N^1 be the $M(1)^+$-submodule of V generated by x^1. Since $(x^1, x^1) = 0$ we see that $M(1)^+ \cap (N^1 \cdot N^1) = 0$. By Lemma 6.3,

$$N^1 \cdot N^1 = \bigoplus_{i=0}^{\infty} (P^{k_i} \cap (N^1 \cdot N^1))$$

and $P^{k_i} \cap (N^1 \cdot N^1)$ is either zero or a direct sum of indecomposable $M(1)^+$-modules with lowest weight k_i^2. Then by Theorem 2.2,

$$P^{k_i} \cap (N^1 \cdot N^1) = 0, \quad \text{or} \ \ k_i^2 = 4k^2. \tag{6.5}$$

This implies that $M^1 \cdot M^1 \subset N^1 \cdot N^1 \subset P^{2k}$. Using the fusion rules from Theorem 2.1 then forces $M^1 \cdot M^1 \cong L(1, 4k^2)$. So the claim holds.

The rest proof of the lemma is quite similar to that of Lemma 5.2 in [DJ2]. We omit it. □

We are now in a position to prove the main result of this paper.

Theorem 6.5. *Let* V *and* k *be as above. Then* V *is isomorphic to the rational vertex operator algebra* V_L^+*, where* $L = \mathbb{Z}\alpha$ *is the rank one positive definite even lattice such that* $(\alpha, \alpha) = 2k^2$.

Proof: By Lemma 6.4, $\dim B_k = 1$, so there exists a non-zero element $F^1 \in V_{k^2}$ such that

$$J_3' F^1 = (4k^4 - k^2)F^1, \ J_m' F^1 = 0, \ L(n)F^1 = 0, \ m \geq 4, n \geq 1,$$

and

$$(F^1, F^1) = 2.$$

Let V_L be the vertex operator algebra associated to the positive even lattice $L = \mathbb{Z}\alpha$ such that $(\alpha, \alpha) = 2k$. Let V^0 be the vertex operator subalgebra of V generated by F^1, J', ω. We first prove that $V^0 \cong V_L^+$.

For $m \in \mathbb{Z}_+$, set

$$E^m = e^{m\alpha} + e^{-m\alpha} \in V_L^+.$$

Then

$$(E^m, E^m) = 2.$$

Denote by N^m the $M(1)^+$-submodule of V_L^+ generated by E^m. Then (see [DN1])

$$N^m = span\{u_n E^m | u \in M(1)^+, n \in \mathbb{Z}\}$$

and

$$N^m \cdot N^n = N^{m-n} \bigoplus N^{m+n}$$

for $m, n \in \mathbb{Z}_+, m \geq n$ where $N^0 = M(1)^+$.

Let W^1 be the $M(1)^+$-submodule of V generated by F^1. Then by Lemma 6.1, W^1 is irreducible. So as $M(1)^+$-modules, $N^1 \cong W^1$. Let σ be the $M(1)^+$-module isomorphism from N^1 to W^1 such that

$$\sigma(E^1) = F^1. \tag{6.6}$$

By Theorem 2.2, Lemma 6.4 and the fact that $(E^1, E^1) = (F^1, F^1)$, we have

$$F_n^1 F^1 = E_n^1 E^1 \in M(1)^+,$$

for $n \geq -2k^2$. Note that

$$(E^1_{-2k^2-1}E^1, E^1_{-2k^2-1}E^1) = (E^1, E^1_{4k^2-1}E^1_{-2k^2-1}E^1)$$
$$= (E^1, \sum_{i=0}^{4k^2-1} \begin{pmatrix} 4k^2-1 \\ i \end{pmatrix} (E^1_j E^1)_{2k^2-2-j} E^1),$$

$$(F^1_{-2k^2-1}F^1, F^1_{-2k^2-1}F^1) = (F^1, F^1_{4k^2-1}F^1_{-2k^2-1}F^1)$$
$$= (F^1, \sum_{i=0}^{4k^2-1} \begin{pmatrix} 4k^2-1 \\ i \end{pmatrix} (F^1_j F^1)_{2k^2-2-j} F^1).$$

This implies that

$$(E^1_{-2k^2-1}E^1, E^1_{-2k^2-1}E^1) = (F^1_{-2k^2-1}F^1, F^1_{-2k^2-1}F^1).$$

Assume that $E^1_{-2k^2-1}E^1 = a_1 E^2 + u^1$, where $0 \neq a_1 \in \mathbb{C}$, $u^1 \in M(1)^+$. Then $F^1_{-2k^2-1}F^1 - u^1$ is either zero or a non-zero primary vector of weight $4k^2$. Since

$$(F^1_{-2k^2-1}F^1 - u^1, F^1_{-2k^2-1}F^1 - u^1) = (a_1 E^1, a_1 E^1),$$

it follows that $F^1_{-2k^2-1}F^1 - u^1 \neq 0$. Let W^2 be the $M(1)^+$-submodule of V generated by $F^1_{-2k^2-1}F^1 - u^1$. Then the isomorphism σ can be extended to the isomorphism from $N^1 \bigoplus N^2$ to $W^1 \bigoplus W^2$ such that

$$\sigma(E^1) = F^1, \ \sigma(E^2) = F^2,$$

where

$$F^2 = \frac{1}{a_1} F^1_{-2k^2-1} F^1 - u^1.$$

So for any $n \in \mathbb{Z}$,

$$\sigma(E^1_n E^1) = (\sigma E^1)_n (\sigma E^1).$$

Following the proof of Lemma 5.7 of [DJ2] and continuing in this way we deduce that $V^0 \cong V_L^+$. The rest of the proof is the same as that of Theorem 5.8 of [DJ2]. □

7 Appendix

Let X^i, $i = 1, 2, \cdots, s$ be the same as in Lemma 4.5. Then we have

Lemma 7.1.

$$\begin{aligned}
&(X^i_4 X^j)_2 X^k \\
=&\frac{1}{2} a^i_{jk} L_{-1} X^i_4 X^i + a^j_{ik}\left(\frac{28}{75} L_{-2} + \frac{11}{30} L^2_{-1}\right) X^j_5 X^j \\
&- \frac{197}{150} a^i_{jk} L_{-1} X^j_4 X^j + \frac{1}{27}\left(\frac{114}{75} a^j_{ik} - 2a^i_{jk}\right) X^{(0)} - 2\sum_{r=1}^{s}\sum_{l=1}^{s} a^r_{jk} a^l_{ir} X^l \\
&+ \frac{114}{75} \sum_{r=1}^{s}\sum_{l=1}^{s} a^r_{ik} a^l_{jr} X^l + a\delta_{jk} X^i + b\delta_{ik} X^j,
\end{aligned}$$

for some $a, b \in \mathbb{C}$.

Proof: By the Jacobi identity, we have

$$\begin{aligned}
&(X^i_4 X^j)_2 X^k \\
=&\sum_{p=0}^{4} (-1)^p \begin{pmatrix} 4 \\ p \end{pmatrix} [X^i_{4-p} X^j_{2+p} - X^j_{6-p} X^i_p] X^k \\
=&[X^i_4 X^j_2 - X^j_6 X^i_0 - 4X^i_3 X^j_3 + 4X^j_5 X^i_1 + 6X^i_2 X^j_4 \\
&- 6X^j_4 X^i_2 - 4X^i_1 X^j_5 + 4X^j_3 X^i_3 - X^j_2 X^i_4] X^k.
\end{aligned}$$

Since

$$X^p_3 X^q = \frac{1}{27} \delta_{pq} X^{(0)} + \sum_{r=1}^{s} a^r_{pq} X^r,$$

we have

$$X^i_3 X^j_3 X^k = X^i_3\left(\frac{1}{27}\delta_{jk} X^{(0)} + \sum_{p=1}^{s} a^p_{jk} X^p\right) = \frac{1}{27}\delta_{jk} X^i_3 X^{(0)} + \frac{1}{27} a^i_{jk} X^{(0)} + \sum_{p,q=1}^{s} a^p_{jk} a^q_{ip} X^q,$$

$$X_3^j X_3^i X^k = X_3^j(\frac{1}{27}\delta_{ik}X^{(0)} + \sum_{p=1}^{s} a_{ik}^p X^p) = \frac{1}{27}\delta_{ik}X_3^j X^{(0)} + \frac{1}{27}a_{ik}^j X^{(0)} + \sum_{p,q=1}^{s} a_{ik}^p a_{jp}^q X^q.$$

Recall Lemma 2.5 and Theorem 2.1. For $p, q \in \{1, 2, \cdots, s\}$,

$$X_2^p X^q = \delta_{pq} u_{p,q} + \sum_{r=1}^{s} a_{pq}^r \frac{1}{2} L(-1) X^r,$$

$$X_1^p X^q = \delta_{pq} v_{p,q} + \sum_{r=1}^{s} a_{pq}^r (\frac{28}{75} L(-2) X^r + \frac{23}{300} L(-1)^2 X^r),$$

$$X_0^p X^q = \delta_{pq} w_{p,q} + \sum_{r=1}^{s} a_{pq}^r (\frac{14}{75} L(-3) X^r + \frac{14}{75} L(-2)L(-1) X^r - \frac{1}{300} L(-1)^3 X^r),$$

where $u_{p,q}, v_{p,q}, w_{p,q} \in L(1,0)$. Note that for $1 \le p \le s$,

$$X_4^p L(-1) = L(-1)X_4^p + 4X_3^p,\ X_5^p L_(-2) = L(-2)X_5^p + 8X_3^p,$$

$$X_5^p L(-1)^2 = (L(-1)X_5^p + 5X_4^p)L(-1) = L(-1)^2 X_5^p + 10L(-1)X_4^p + 20X_3^p,$$
$$X_6^p L(-3) = L(-3)X_6^p + 12X_3^p,\ X_6^p L(-1) = L(-1)X_6^p + 6X_5^p,$$

$$\begin{aligned} X_6^p L(-2)L(-1) &= (L(-2)X_6^p + 9X_4^p)L(-1) \\ &= L(-2)L(-1)X_6^p + 6L(-2)X_5^p + 9L(-1)X_4^p + 36X_3^p, \end{aligned}$$

$$X_6^p L(-1)^3 = L(-1)^3 X_6^p + 18L(-1)^2 X_5^p + 90L(-1)X_4^p + 120X_3^p.$$

Then we have

$$\begin{aligned} &X_4^i X_2^j X^k \\ =&\delta_{jk} X_4^i u_{j,k} + \sum_{p=1}^{s} a_{jk}^p \frac{1}{2} X_4^i L(-1) X^p \\ =&\delta_{jk} X_4^i u_{j,k} + \sum_{p=1}^{s} a_{jk}^p \frac{1}{2} (L(-1)X_4^i + 4X_3^i) X^p \\ =&\delta_{jk} X_4^i u_{j,k} + \frac{1}{2} a_{jk}^i L(-1) X_4^i X^i + \frac{2}{27} a_{jk}^i X^{(0)} + 2\sum_{p,q=1}^{s} a_{jk}^p a_{ip}^q X^q, \end{aligned}$$

$$\begin{aligned} &X_4^j X_2^i X^k \\ =&\delta_{ik} X_4^j u_{i,k} + \frac{1}{2} a_{ik}^j L(-1) X_4^j X^j + \frac{2}{27} a_{ik}^j X^{(0)} + 2\sum_{p,q=1}^{s} a_{ik}^p a_{jp}^q X^q, \end{aligned}$$

$$
\begin{aligned}
&X_5^jX_1^iX^k\\
=&\delta_{ik}X_5^jv_{i,k}+X_5^j\sum_{p=1}^s a_{ik}^p(\frac{28}{75}L(-2)X^p+\frac{23}{300}L(-1)^2X^p)\\
=&\delta_{ik}X_5^jv_{i,k}+\sum_{p=1}^s a_{ik}^p\frac{28}{75}(L(-2)X_5^j+8X_3^j)X^p\\
&+\frac{23}{300}\sum_{p=1}^s a_{ik}^p(L(-1)^2X_5^j+10L(-1)X_4^j+20X_3^j)X^p\\
=&\delta_{ik}X_5^jv_{i,k}+\frac{28}{75}a_{ik}^jL(-2)X_5^jX^j+\frac{23}{300}a_{ik}^jL(-1)^2X_5^jX^j\\
&+\frac{23}{30}a_{ik}^jL(-1)X_4^jX^j+\frac{113}{25\times 27}a_{ik}^jX^{(0)}+\frac{113}{25}\sum_{p,q=1}^s a_{ik}^pa_{jp}^qX^q,
\end{aligned}
$$

$$
\begin{aligned}
&X_6^jX_0^iX^k\\
=&\delta_{ik}X_6^jw_{i,k}+X_6^j\sum_{p=1}^s a_{ik}^p(\frac{14}{75}L(-3)X^p+\frac{14}{75}L(-2)L(-1)X^p-\frac{1}{300}L(-1)^3X^p)\\
=&\delta_{ik}X_6^jw_{i,k}+\sum_{p=1}^s a_{ik}^p\frac{14}{75}(L(-3)X_6^j+12X_3^j)X^p\\
&+\frac{14}{75}\sum_{p=1}^s a_{ik}^p(L(-2)L(-1)X_6^j+6L_{-2}X_5^j+9L(-1)X_4^j+36X_3^j)X^p\\
&-\frac{1}{300}\sum_{p=1}^s a_{ik}^p(L(-1)^3X_6^j+18L(-1)^2X_5^j+90L(-1)X_4^j+120X_3^j)X^p\\
=&\delta_{ik}X_6^jw_{i,k}+\frac{28}{25}a_{ik}^jL(-2)X_5^jX^j-\frac{3}{50}a_{ik}^jL(-1)^2X_5^jX^j\\
&+\frac{69}{50}a_{ik}^jL(-1)X_4^jX^j+\frac{214}{25\times 27}a_{ik}^jX^{(0)}+\frac{214}{25}\sum_{p,q=1}^s a_{ik}^pa_{jp}^qX^q.
\end{aligned}
$$

Note that

$$
X_2^iX_4^jX^k, X_1^iX_5^jX^k, X_4^i\delta_{jk}u_{j,k}\in\mathbb{C}(\delta_{jk}X^i),
$$
$$
X_2^jX_4^iX^k, X_4^j\delta_{ik}u_{i,k}, X_5^j\delta_{ik}v_{i,k}, X_6^j\delta_{ik}w_{i,k}\in\mathbb{C}(\delta_{ik}X^j).
$$

Then it is easy to check that the lemma holds. □

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